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The structure of the ground ring in critical W_3 gravity

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Abstract

By explicit calculation, I determine the structure of the ground ring of the critical W_3 gravity and show that there is an $su(3)$ invariant quadratic relation among the six basic elements. By using this result, I also construct some discrete physical states of the critical W_3 gravity.

The study of W gravity is surely an interesting extension of the usual two dimensional gravity coupled with matter. The matter part is the minimal W-matter. Particularly interesting is the borderline case where the matter part has an integral central charge which is equal to the rank of the associated Lie algebra. (I will call this case the critical W gravity.) The study of the simplest case [1–6], the $D = 2$ string theory, has revealed a lot of interesting structures: the existence of a ground ring [3] and the infinite dimensional Lie algebra generated by ghost number 0 currents [3–5]. I will report in this letter some results on the extension of these structures to the critical W_3 gravity.

Several papers [7–12] have studied the cohomology problem of W gravity. Nevertheless the results are still not complete and explicit. The methods used often turn out to be quite transendential, or by using computer to do most of the calculations, the results are quite messy and are of little use. Especially important is what is the analogous ground ring structure and what is the infinite dimensional Lie algebra in these W gravity theories. Many conjectures exist in the literature and very few proofs and explicit results are offered. Here I will study the structure of the ground ring in critical W_3 gravity by explicit calculation. Although most of the calculations are also done by computer [13,14], I have try my best to give the results in their simplest analytic forms. I will show that the ground ring in critical W_3 gravity is not a free polynomial ring of 6 elements. This ring is actually a polynomial ring of 6 elements module

one quadratic relation. This quadratic relation is $su(3)$ invariant. By exploiting the property of this ground ring, I also construct some discrete physical states. Partial results have also been obtained for the algebra generated by the discrete states and the generalization to other critical W gravities, but I will not report these results here.

As usual I deal with only one chiral sector and restrict my attention to the critical case. I will speak only the prime physical states or the relative physical states in a generalized relative cohomology. For the pure W_3 gravity, the complete physical states have been obtained in [7] but there is no interesting symmetry structures, just as in the case with the usual pure gravity (i.e. W_2 gravity). For the W_3 gravity coupled with (the W_3) minimal matter, the study and enumeration of all the physical states is complicated by the problem of decoupling all the null matter states, see [12] and refernces therein. The critical W_3 gravity is easier and is also more interesting. This theory has been studied in [8,11].

The basic fields of the critical W_3 gravity are the two free matter fields $X_1(z)$ and $X_2(z)$, two Liouville fields $\phi_1(z)$ and $\phi_2(z)$, two pairs of ghost anti-ghost fields $(b(z), c(z))$ and $(\beta(z), \gamma(z))$ associated with the spin 2 and 3 generators of the W_3 algebra. From these fields we can construct the following stress-energy tensors and two spin-3 generators:

$$\begin{aligned}
T_X &= -\frac{1}{2}(\partial_z X_1(z))^2 - -\frac{1}{2}(\partial_z X_2(z))^2, && (1)\\
T_\phi &= -\frac{1}{2}(\partial_z \phi_1(z))^2 - -\frac{1}{2}(\partial_z \phi_2(z))^2 + \sqrt{2}\partial_z^2\phi_1(z) + \sqrt{6}\partial_z^2\phi_2(z), && (2)\\
T_{bc} &= \quad 2\,\partial_z c(z)\, b(z) + c(z)\,\partial_z b(z), && (3)\\
T_{\beta\gamma} &= \quad 3\,\partial_z\gamma(z)\,\beta(z) + 2\gamma(z)\,\partial_z\beta(z), && (4)
\end{aligned}$$

and

$$\begin{aligned}
W_X &= \quad \frac{i}{6}\Big(3\,(\partial_z X_1(z))^2 - (\partial_z X_2(z))^2\Big)\,\partial_z X_2(z), && (5)\\
W_\phi &= \quad \frac{i}{24}\Big(3\,(\partial_z\phi_1(z))^2\,\partial_z\phi_2(z) - (\partial_z\phi_2(z))^3 + 3\sqrt{6}\partial_z\phi_2(z)\,\partial_z^2\phi_2(z)\\
&\quad - 3(\sqrt{6}\,\partial_z\phi_1(z) + 2\sqrt{2}\,\partial_z\phi_2(z))\partial_z^2\phi_1(z) + 6\sqrt{3}\,\partial_z^3\phi_1(z) - 6\,\partial_z^2\phi_2(z)\Big). && (6)
\end{aligned}$$

The two pairs (T_X, W_X) and (T_ϕ, W_ϕ) satisfy the nonlinear W_3 algebra [16] with central charge $c = 2$ and 98 respectively. The combined matter-Liouville stress-energy tensor $T_{X\phi} = T_X + T_\phi$ then has central charge 100 which is the critical central charge of the W_3 algebra [17]. It is then possible to have a nilpotent BRST for this matter-Liouville-ghost-antighost system. Actually

two different nilpotent BRST charges are known [17–19]. The one obtained in [18,19] is better suited for the study of the critical W_3 gravity. This is because the standard physical tachyon-like states have a simple form with respect to this BRST charge. We will use the following BRST charge for all our explicit calculation, although the results obtained don't depend on the exact form of the BRST charge. The BRST charge Q is the contour integration of the BRST current $j(z)$: $Q \equiv \oint_0 [\mathrm{dz}]\mathrm{j(z)}$ and

$$
\begin{aligned}
j(z) = & c(z)\left(T_X(z) + T_\phi(z) + \partial_z c(z)\, b(z) + T_{\beta\gamma}(z)\right) \\
& + \sqrt{6}\,\gamma(z)\Big(W_X(z) + 4i\, W_\phi(z)\Big) + 3\left(T_X(z) - T_\phi(z)\right) b(z)\,\gamma(z)\,\partial_z\gamma(z) \\
& - \frac{3}{2}\Big(2\,\gamma(z)\,\partial_z^3\gamma(z) - 3\,\partial_z\gamma(z)\,\partial_z^2\gamma(z)\Big)\, b(z). \qquad (7)
\end{aligned}
$$

Here we have rescaled γ and β by a factor $\sqrt{3}$ to avoid some unnecessary $\sqrt{3}$ factors in the expression of ground ring states (see below), comparing with [8].

Denoting the six different Weyl transformations of the Weyl group of the $su(3)$ Lie algebra as σ_i, $i = 0, \cdots, 5$, the six tachyon-like physical states are given as follows

$$
V_{p_X, p_\phi}(z) = c(z)\,\gamma(z)\,\partial_z\gamma(z)\, e^{i\,\sigma_i\,(p_\phi - 2\,\rho)\cdot X(z) + p_\phi\cdot\phi(z)}, \qquad i = 0, \cdots, 5. \quad (8)
$$

For generic momentum p_ϕ, the above tachyon-like states are the only physical states with two arbitrary parameters. Later we will identify some one parameter continuous physical states.

Ground ring was introduced in the study of $D = 2$ string theory by Witten [3]. However this concept has more general usage and can be extended to the study of other theories. The ground ring for W_3 gravity has been discussed in [8] and explicit formulas are also given for the basic elements. However, little was known about the propertites of the ring structure and their applications little exploited. Let us first recall some basics about the ground ring in $D = 2$ string theory. In $D = 2$ string theory, there exist two basic physical states (with ghost number 0) which form the two dimensional multiplet, the lowest nontrivial representation of the $su(2)$ algebra. Explicitly these two states are given as follows:

$$
x = (c\,b + S_1(iX^-))\, e^{i\,X^+}, \qquad (9)
$$
$$
y = (c\,b + S_1(-iX^+))\, e^{-i\,X^+}, \qquad (10)
$$

where $X^{\pm} = \frac{1}{\sqrt{2}}(X \pm i\,\phi)$ and S_1 is the first Schur polynomial[1]. The ground ring is just the **free** polynomial ring in x and y if we define the multiplication as the usual normal ordering at the same point module BRST exact terms. The states with a fixed momentum p_ϕ form a $2\,j\!+\!1$ dimensional representation of the $su(2)$ algebra and these states corresponds to the ring elements with a fixed degree $2\,j\!+\!1$, where j is an integer or half integer. In critical W_3 gravity, we need two basic representations of the $su(3)$ algebra which are denoted as $\mathbf{3}$ (with highest weight λ_1) and $\bar{\mathbf{3}}$ (with highest weight λ_2) or $\{1,0\}$ and $\{0,1\}$ in the notation of Dykin index. We also denote the elements of $\mathbf{3}$ as x_i, $i=1,2,3$ and those of $\bar{\mathbf{3}}$ as y_i, $i=1,2,3$. These states have the following general form:

$$
\begin{aligned}
x_i &= (\text{something of ghost number } 0) \times e^{i\,m_i\cdot X - \lambda_2\cdot\phi}, && (11)\\
y_i &= (\text{something of ghost number } 0) \times e^{-i\,m_i\cdot X - \lambda_1\cdot\phi}, && (12)
\end{aligned}
$$

where m_i's are the three weights of the representation $\mathbf{3}$ (with highest weight λ_1). Here we have used the fact that the weights of $\bar{\mathbf{3}}$ can be obtained from the weights of $\mathbf{3}$ by multiplying -1. We will also denote x_i as $O^{\lambda_1}_{m_i}$ and y_i as $O^{\lambda_2}_{-m_i}$ where the superscript denotes the representation and the subscript denotes the X-momentum. The explicit expression of these states can be obtained by using computer. We have take a little time to write the resulting expression in the following form:

$$
\begin{aligned}
x_1 \equiv O^{\lambda_1}_{\lambda_1} = \Big(& (c\,b\,\gamma\,\beta + c\,b' + \frac{3}{2}\gamma\,\beta + \frac{7}{2}\gamma'\,\beta - 7\,b\,b'\,\gamma\,\gamma' \\
& - \frac{5}{2}b\,\gamma\,\gamma'\,\beta - \frac{21}{4}b\,\gamma'' - \frac{1}{2}c\,\beta + \frac{1}{2}c\,b\,b'\,\gamma - 2\,b'\,\gamma' - \frac{3}{4}b''\,\gamma) \\
& + \frac{3}{2}(c\,b + \gamma\,\beta + 3\,b\,\gamma' + b'\,\gamma)\,S_1(i\lambda_1\cdot X) + \frac{3}{2}(\gamma\,\beta - 2\,b\,\gamma')\,\lambda_2\cdot\phi' \\
& + 3\,(1+b\,\gamma)(S_2(i\lambda_1\cdot X) + \frac{1}{2}S_1(i\lambda_1\cdot X)\,\lambda_2\cdot\phi') \\
& + 3\,(1-2\,b\,\gamma)(-\frac{1}{2}(\lambda_1-\lambda_2)\cdot\phi'\,\lambda_1\cdot\phi' + \lambda_1\cdot\phi'')\Big)\,e^{i\lambda_1\cdot X - \lambda_2\cdot\phi}, \qquad (13)
\end{aligned}
$$

where S_i's are Schur polynomials. The two other states in $\mathbf{3}$ can be obtained by $su(3)$ action. The resulting expression is the same as above but with $S_i(i\lambda_1\!\cdot\! X)$ change to $S_i(i\,m_i\!\cdot\! X)$ in accordance with the exponetial factor. The expression for the states y_i in $\bar{\mathbf{3}}$ can also be obtained from (13) by noting the following

[1] Explicitly we have:

$$S_k(i\delta\cdot X(w))e^{i\delta\cdot X(w)} = \frac{1}{k!}\,\partial^k_w(e^{i\delta\cdot X(w)}).$$

transformation of the BRST charge:

$$\gamma \rightarrow -\gamma, \qquad \beta \rightarrow -\beta, \qquad X \rightarrow -X, \qquad \lambda_1 \cdot \phi \leftrightarrow \lambda_2 \cdot \phi. \tag{14}$$

This transformation transforms one nontrivial physical states into another nontrivial physical states. Explicitly we have:

$$\begin{aligned} y_1 \equiv O^{\lambda_2}_{-\lambda_1} = \Big(& (c\, b\, \gamma\, \beta + c\, b' + \frac{3}{2}\, \gamma\, \beta + \frac{7}{2}\, \gamma'\, \beta - 7\, b\, b'\, \gamma\, \gamma' \\ & + \frac{5}{2}\, b\, \gamma\, \gamma'\, \beta + \frac{21}{4}\, b\, \gamma'' + \frac{1}{2}\, c\, \beta - \frac{1}{2}\, c\, b\, b'\, \gamma + 2\, b'\, \gamma' + \frac{3}{4}\, b''\, \gamma) \\ & + \frac{3}{2}\, (c\, b + \gamma\, \beta - 3\, b\, \gamma' - b'\, \gamma)\, S_1(-i\lambda_1 \cdot X) + \frac{3}{2} (\gamma\, \beta + 2\, b\, \gamma')\, \lambda_1 \cdot \phi' \\ & + 3\, (1 - b\, \gamma) (S_2(-i\lambda_1 \cdot X) + \frac{1}{2}\, S_1(-i\lambda_1 \cdot X)\, \lambda_1 \cdot \phi') \\ & + 3\, (1 + 2\, b\, \gamma) (\frac{1}{2}\, (\lambda_1 - \lambda_2) \cdot \phi'\, \lambda_2 \cdot \phi' + \lambda_2 \cdot \phi'') \Big)\, e^{-i\lambda_1 \cdot X - \lambda_1 \cdot \phi}. \end{aligned} \tag{15}$$

The other two y_i's can be obtained from this one by $su(3)$ action. This changes only the arguments of the the Schur polynomials and the exponential factor.

Similar to $D = 2$ string theory, the ground ring of the critical W_3 gravity is the polynomial ring in six elements x_i and y_i, $i = 1, 2, 3$. The difference is that this ring is **not free**, i.e. there is a polynomial relation between the six basic elements x_i and y_j:

$$x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 = 0. \tag{16}$$

This is equivalent to say that the tensoring of $\mathbf{3}$ and $\bar{\mathbf{3}}$ gives only $\mathbf{8}$. The state which could give rise to the trival representation $\mathbf{1}$ is proportinal to the left hand side of eq. (16), but it is vanishing. It is quite difficult to prove the above equation directly. I will talk about the proof in a moment, but let us first explore the the consequence of the above equation. By taking tensor product of the two basic representations we can obtain any representations. Because of eq. (16), the structure of tensor product is greatly simplified. We have only the following rules:

$$\{n_1, n_2\} \otimes \{m_1, m_2\} = \{n_1 + m_1, n_2 + m_2\}, \tag{17}$$

i.e. the products of the states in two highest representations with highest weights Λ_1 and Λ_2 give only states which form another highest weight representation with highest weight $\Lambda = \Lambda_1 + \Lambda_2$. Presumably this procedure gives all the physical states with ghost number 0 (more about this point at the end of this paper). So we conclude that only for $p_\phi = -(n_2\, \lambda_1 + n_1\, \lambda_2)$,

(n_1, n_2: non-negative integers) there exist ghost number 0 relative physical states. These states form a highest weight representation of $su(3)$ with highest weight $\Lambda = (n_1\,\lambda_1 + n_2\,\lambda_2)$ and their expression can be obtained from the highest state $O^\Lambda_\Lambda \equiv (O^{\lambda_1}_{\lambda_1})^{n_1} \cdot (O^{\lambda_2}_{\lambda_2})^{n_2}$ by repeatedly using $su(3)$ actions.

Defining two operators

$$\begin{aligned} a_+(n,m) &\equiv [Q, -i\,(n+2)\,\alpha_1 \cdot X + m\,\alpha_2 \cdot \phi], \qquad (18)\\ a_-(n,m) &\equiv [Q, -i\,(n+2)\,\alpha_2 \cdot X + m\,\alpha_2 \cdot \phi], \qquad (19) \end{aligned}$$

one can prove that the multiplication of $a_+(n_1, m_1)$ and $a_-(n_1, m_2)$ with $O^\Lambda_{\lambda, q(\lambda)}$, having $\Lambda = n_1\,\lambda_1 + n_2\,\lambda_2$ and $\lambda = m_1\,\lambda_1 + m_2\,\lambda_2$, give rise to new nontrivial relative states with ghost number 1. Actually these states belong to the irreducible representation $\Lambda + \rho$ and its highest weight state is

$$\begin{aligned} \bar{O}^{\Lambda+\rho}_{\Lambda+\rho} &\equiv [E_\rho, a_+] \cdot O^\Lambda_\Lambda \\ &= [E_\rho, a_+ \cdot O^\Lambda_\Lambda]. \qquad (20) \end{aligned}$$

This gives some physical states of ghost number 1 with Liouville momentum $p_\phi = -(n_2\,\lambda_1 + n_1\,\lambda_2)$. These states don't exhaust all the discrete phsysical states in the critical W_3 gravity with ghost number 1.[2]

The physical states with ghost number 2 can be similarly constructed and we have the following results: the physical states (having ghost number 2 and fixed Liouville momentum $p_\phi = -(n_2\,\lambda_1 + n_1\,\lambda_2)$ form two irreducible representations $\Lambda + 3\,\lambda_1$ and $\Lambda + 3\,\lambda_2$ which are generated by the following two highest weight states:

$$\begin{aligned} \underline{Y}^{\Lambda+3\,\lambda_1}_{\Lambda+3\,\lambda_1} &\equiv [E_{\alpha_1}, [E_\rho, a_+]] \cdot O^\Lambda_\Lambda \\ &= [E_{\alpha_1}, [E_\rho, a_+ \cdot O^\Lambda_\Lambda]], \qquad (21)\\ \underline{Y}^{\Lambda+3\,\lambda_2}_{\Lambda+3\,\lambda_2} &\equiv [E_{\alpha_2}, [E_\rho, a_+]] \cdot O^\Lambda_\Lambda \\ &= [E_{\alpha_2}, [E_\rho, a_+ \cdot O^\Lambda_\Lambda]]. \qquad (22) \end{aligned}$$

Actually the physical states at the boundary of the above representations can also be promoted to one parameter continuous physical states, just like the boundary states (with ghost number 1) in $D = 2$ string theory are actually tachyon states. Here in critical W_3 gravity, the ghost number of continuous states can be ranged from 2 to 4. (If we take into account also the absolute states, the range is from 2 to 6.)

[2] I would like to thank K. Pilch and P. Bouwknegt for pointing out this to me.

Finally let us also mention that the discrete states with ghost number 3 are in representation $\Lambda + 2\,\rho$ and can be generated by the following highest weight state

$$\begin{aligned} Y^{\Lambda+2\,\rho}_{\Lambda+2\,\rho} \equiv\ & [E_{\alpha_2},[E_{\alpha_1},[E_\rho,a_+]]]\cdot O^\Lambda_\Lambda \\ =\ & [E_{\alpha_2},[E_{\alpha_1},[E_\rho,a_+\cdot O^\Lambda_\Lambda]]] \propto c\,\gamma\,\gamma'\,e^{i\,(\Lambda+2\,\rho)\cdot X-(n_2\,\lambda_1+n_1\,\lambda_2)\cdot\phi}. \end{aligned} \tag{23}$$

Presumably there exists no relative discrete states with ghost number 4 or greater and having the above mentioned Liouville momentum.

In the above, we have studied only the physical states having the same Liouville momentum as the ground ring elements. There are about five copies of these states having other Liouville momentum and ghost number, just like there are minus states in addition to plus states in $D = 2$ string theory. A complete classificaton of all the physical states for the critical W_3 gravity is not available. Now let me return to the proof of eq. (16).

In order to prove eq. (16), we use the following theorem:

- If the current $V^{(1)}(w)$ of a BRST invariant state $V(w)$ is zero module BRST exact and total derivative terms, i.e. $V^{(1)}(w)$ can be written as follows:

$$V^{(1)}(w) = \{Q,\eta^{(1)}] + \partial_w\,\eta^{(2)}(w), \tag{24}$$

  then we have

$$\partial_w\left(V(w) - \{Q,\eta^{(2)}(w)]\right) = 0, \tag{25}$$

  i.e. $V(w)$ is either a BRST exact state or proportional to the trivial BRST invariant state 1.

This theorem can be proved easily by using contour deformation in conformal field theory. The only requirement is that the BRST charge should be written as a contour integration of a dimension 1 current and that the (anti-) commutator of Q with $b(w)$ gives the total stress energy tensor. This is satisfied by the BRST charge given in (7).

By using this theorem, one proves eq. (16) if one can show that the corresponding current of the left hand side is the form of (24). This is not so difficult with our explicit expression for x_i and y_i given in eqs. (13) and (15). However the current

$$j_i \equiv \oint_w [\,\mathrm{dz}]\,\mathrm{b(z)}\,(\mathrm{x_i}\cdot\mathrm{y_i})(\mathrm{w})$$

$$= \oint_w [\,\mathrm{dz}](\mathrm{b(z)\,x_i(w)}) \cdot \mathrm{y_i(w)} + \mathrm{x_i(w)} \cdot \oint_{\mathrm{w}} [\,\mathrm{dz}]\mathrm{b(z)\,y_i(w)}, \tag{26}$$

still contains something over 100 terms. After subtracting all the total derivative and BRST exact terms, I found that there are only two independent j_i, i.e. there is a linear relation among the three j_i's:

$$j_1 + j_2 + j_3 = 0. \tag{27}$$

This proves (16). It may be helpful to display some terms of j_i's here to see how eq. (27) is satisfied. We have

$$\begin{aligned} j_1 = \Big(& \frac{9}{2}\,\partial^2 \tilde{X}_1\,\rho\cdot\partial\phi\, b\,\gamma\,\beta + (3\,\partial\tilde{X}_1\,\rho\cdot\partial\phi - \frac{15}{2}\,\partial^2\tilde{X}_1)\, b\,\gamma\,\beta' \\ & + \frac{27}{4}\,\partial\tilde{X}_1\,\rho\cdot\partial\phi\, b\,\gamma'\,\beta - \frac{45}{4}(2\,b\,\gamma'\,\beta' + 2\,b\,\gamma\,\beta'' + b\,\gamma''\,\beta)\partial\tilde{X}_1 \\ & - \Big(9\,(\lambda_1 - \lambda_2)\cdot\partial\phi\,\partial\tilde{X}_1 + \frac{15}{4}\,(\partial\tilde{X}_1)^2 - \frac{15}{2}\partial\tilde{X}_1\,\partial\tilde{X}_2 \\ & - \frac{15}{2}\,(\partial\tilde{X}_2)^2\Big)\, b\, b'\,\gamma' + (\cdots)\, b\, b'\,\gamma + (\cdots)\,\beta + (\cdots)\, b\Big)\, e^{-i\rho\cdot\phi}, \end{aligned} \tag{28}$$

where $\tilde{X}_1 = i\,\lambda_1\cdot X$ and $\tilde{X}_2 = -i\,\lambda_2\cdot X$. Notice that the above expression changes to $-j_1$ under the transformation (14). This is understandable because $x_1\!\cdot\! y_1$ is defined up to a sign and $x_1\!\cdot\! y_1 = \pm y_1\!\cdot\! x_1$. The other two currents j_2 and j_3 can be obtained from the above expression by the following substitutions:

$$\tilde{X}_1 \to \tilde{X}_2, \qquad\qquad \tilde{X}_2 \to -\tilde{X}_1 - \tilde{X}_2, \tag{29}$$

or

$$\tilde{X}_1 \to -\tilde{X}_1 - \tilde{X}_2, \qquad\qquad \tilde{X}_2 \to \tilde{X}_1. \tag{30}$$

The terms linear in $\tilde{X}_1$ or $\tilde{X}_2$ cancell outomatically in eq. (27). We have also proved that other terms cancell with each other.

Actually one can argue that there must exist some relations such as eq. (16) from other sources [11]. One way to prove this relation is to have some similar existence and unique theorem about the spectrum of physical states in W_3 gravity. In [11], some theorems and conjectures have been given. It's quite difficult to understand their arguments (at least for me). Our explicit calculation actually solved the cohomology problem of critical W_3 gravity by explicit construction. We found that by taking tensor product one gets only once each representation, i.e. for a fixed Liouville momentum, there is only one irreducible finite-dimensional $su(3)$ ground ring module. As noted in [11], there

may also exist some other ground ring elements which cann't be obtained by taking tensor product of the two basic representations (this may appear in the interior of the Weyl chamber). To settle this question explicitly, I have compute all the possible physical states of ghost number 0 and momentum $p_\phi = -(\lambda_1 + \lambda_2)$ by brute force by using computer. There are only 8 independent non-trivial physical states. This shows that there are no other physical states other that those obtained from tensor product of the two basic representations. This result has also been prove by K. Pich and P. Bouwknegt by using a different programme to do the explicit computations.

It's quite difficult to extend our method to other more complicated critical W_n gravity theories. Nevertheless the results can be easily generalized to other critical W gravities associated with any simply laced Lie group. We have: *the ground ring in any critical W gravity is the polynomial ring of all the states of the basic representations module relations which would give non-maximal representations when taking tensor product. In other words, the ground ring exists and its structure is such that the product of any two highest weight representations gives only one highest weight representation.* For a fixed Liouville momentum, there is only one irreducible finite-dimensional ground ring module. I hope there would be a different way to arrive at (or to disprove) these conjectures.

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